

Mathematical Tools for Physics

In this module, you will get a basic overview of many concepts of mathematics that are extensively used in physics. Note that ALL these topics will be taught to you in detail in mathematics at a later stage. The introductory level mathematics given here will be sufficient to make you comfortable with the mathematics used in Physics in class 11th and 12th. You are strongly advised to master all the concepts and formulae given here. DO NOT try to get into unnecessary mathematical details at this stage as they will anyway be covered in mathematics at an appropriate time.

TRIGONOMETRY

Section - 1

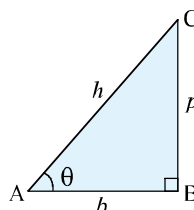
Recall from your previous class that for a right angled triangle ABC , AC is called hypotenuse (h), AB is called the base (b) and BC is called the perpendicular (p) or height. The trigonometric ratios of the angle θ as shown are defined as

$$\sin \theta = \frac{p}{h}; \cos \theta = \frac{b}{h}; \tan \theta = \frac{p}{b}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{h}{p}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{h}{b}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{b}{p}$$



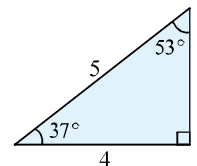
The angle θ is generally measured in degrees but the SI unit of angle is radian. 2π radian is equivalent to 360° (it is the angle subtended by a circle at its centre).

$$\text{Hence } 30^\circ = \frac{\pi}{6} \text{ rad}, 45^\circ = \frac{\pi}{4} \text{ rad}, 60^\circ = \frac{\pi}{3} \text{ rad}, 90^\circ = \frac{\pi}{2} \text{ rad}, 180^\circ = \pi$$

You have also learnt about the values of trigonometric ratios for angles $0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$. The table below lists these values along with two more commonly encountered angles 37° and 53° .

θ	0°	30°	45°	60°	90°	37°	53°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{3}{5}$	$\frac{4}{5}$
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$\frac{4}{5}$	$\frac{3}{5}$
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined	$\frac{3}{4}$	$\frac{4}{3}$
$\cot \theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$\frac{4}{3}$	$\frac{3}{4}$
$\sec \theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined	$\frac{5}{4}$	$\frac{5}{3}$
$\operatorname{cosec} \theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{5}{3}$	$\frac{5}{4}$

Note : For a triangle with the angles as $90^\circ, 37^\circ$ and 53° , the sides are in the ratio $3 : 4 : 5$. You can use this to find the trigonometric ratios of 37° and 53° .

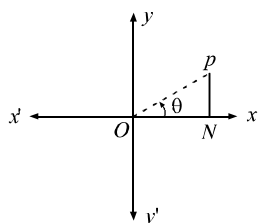
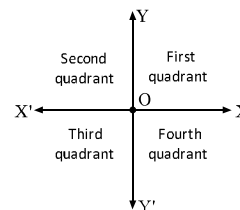


For finding trigonometric ratios of angles greater than 90° , the following sign convention is used.

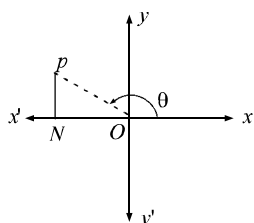
Four Quadrants and Sign Conventions :

Consider two mutually perpendicular lines intersecting at O. These two mutually perpendicular lines divide the plane into four parts called quadrants (see figure).

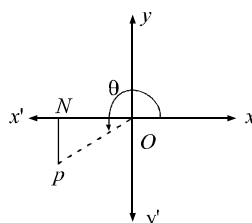
We are interested in finding the various trigonometric ratios for a given angle θ ($0 \leq \theta \leq 360^\circ$). The angle θ is measured from +x axis in anticlockwise sense.



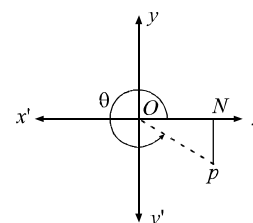
(a) θ in first quadrant



(b) θ in second quadrant



(c) θ in third quadrant



(d) θ in fourth quadrant

To determine the sign of a trigonometric ratio in any quadrant, OP is taken as positive in all the four quadrants while ON and NP are taken as positive or negative depending on the quadrant in which θ falls. Accordingly,

- (i) In the first quadrant, all trigonometrical ratios are positive.

(As ON is +ve and NP is +ve)

- (ii) In the second quadrant, only $\sin \theta$ and $\operatorname{cosec} \theta$ are positive.

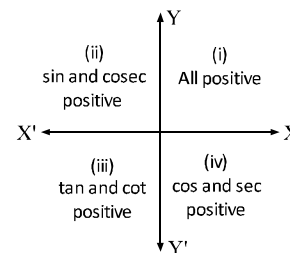
(As ON is -ve and NP is +ve)

- (iii) In the third quadrant, only $\tan \theta$ and $\cot \theta$ are positive.

(As ON and NP are both -ve)

- (iv) In the fourth quadrant, only $\cos \theta$ and $\sec \theta$ are positive.

(As ON is +ve and NP is -ve)



The value of $\sin \theta$ and $\cos \theta$ are such that $-1 \leq \sin \theta \leq 1$ and $-1 \leq \cos \theta \leq 1$. But $\tan \theta$ and $\cot \theta$ can take any real value.

Trigonometric Ratios of Allied Angles :

The angles whose sum or difference with angle θ is zero or a multiple of 90° are called angles allied to θ . Commonly used trigonometric ratios of some allied angles are given below.

1. $\sin(-\theta) = -\sin \theta$, $\operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta$

$$\cos(-\theta) = \cos \theta, \sec(-\theta) = \sec \theta$$

$$\tan(-\theta) = -\tan \theta, \cot(-\theta) = -\cot \theta$$

Note : As angle $-\theta$ lies in the fourth quadrant, only $\cos \theta$ and $\sec \theta$ are positive, e.g.,

$$\sin(-30^\circ) = -\sin 30^\circ \text{ and } \cos(-30^\circ) = +\cos 30^\circ.$$

2. $\sin(90^\circ - \theta) = \cos \theta$, $\operatorname{cosec}(90^\circ - \theta) = \sec \theta$

$$\cos(90^\circ - \theta) = \sin \theta, \sec(90^\circ - \theta) = \operatorname{cosec} \theta$$

$$\tan(90^\circ - \theta) = \cot \theta, \cot(90^\circ - \theta) = \tan \theta$$

Note : As angle $(90^\circ - \theta)$ lies in the first quadrant, all trigonometric ratios are positive, e.g.,

$$\sin 30^\circ = \sin(90^\circ - 60^\circ) = \cos 60^\circ.$$

3. $\sin(90^\circ + \theta) = \cos \theta$, $\operatorname{cosec}(90^\circ + \theta) = \sec \theta$
 $\cos(90^\circ + \theta) = -\sin \theta$, $\sec(90^\circ + \theta) = -\operatorname{cosec} \theta$
 $\tan(90^\circ + \theta) = -\cot \theta$, $\cot(90^\circ + \theta) = -\tan \theta$

Note : As angle $(90^\circ + \theta)$ lies in second quadrant, only $\sin \theta$ and $\operatorname{cosec} \theta$ are positive, e.g.,
 $\sin 120^\circ = \sin(90^\circ + 30^\circ) = \cos 30^\circ$.

4. $\sin(180^\circ - \theta) = \sin \theta$, $\operatorname{cosec}(180^\circ - \theta) = \operatorname{cosec} \theta$
 $\cos(180^\circ - \theta) = -\cos \theta$, $\sec(180^\circ - \theta) = -\sec \theta$
 $\tan(180^\circ - \theta) = -\tan \theta$, $\cot(180^\circ - \theta) = -\cot \theta$

Note : As angle $(180^\circ - \theta)$ lies in the second quadrant, only $\sin \theta$ and $\operatorname{cosec} \theta$ are positive, e.g.,
 $\sin 150^\circ = \sin(180^\circ - 30^\circ) = \sin 30^\circ$

5. $\sin(180^\circ + \theta) = -\sin \theta$, $\operatorname{cosec}(180^\circ + \theta) = -\operatorname{cosec} \theta$
 $\cos(180^\circ + \theta) = -\cos \theta$, $\sec(180^\circ + \theta) = -\sec \theta$
 $\tan(180^\circ + \theta) = \tan \theta$, $\cot(180^\circ + \theta) = \cot \theta$

Note : As angle $(180^\circ + \theta)$ lies in the third quadrant, only $\tan \theta$ and $\cot \theta$ are positive, e.g.,
 $\tan 210^\circ = \tan(180^\circ + 30^\circ) = \tan 30^\circ$.

Illustration - 1 Determine the values of :

- (i) $\sin 120^\circ$ (ii) $\sin 240^\circ$ (iii) $\sin(-30^\circ)$ (iv) $\sin 300^\circ$

SOLUTION :

- (i) This angle lies in the second quadrant. In the second quadrant, sine is positive. Therefore, $\sin 120^\circ =$ some positive value.

$$\sin 120^\circ = \sin(90^\circ + 30^\circ) = \cos 30^\circ = \sqrt{3}/2$$

- (ii) This angle lies in the third quadrant. So $\sin 240^\circ$ should be negative.

$$\sin 240^\circ = \sin(180^\circ + 60^\circ) = -\sin 60^\circ = -\sqrt{3}/2$$

- (iii) This angle lies in the fourth quadrant. So $\sin(-30^\circ)$ should be negative.

$$\sin(-30^\circ) = -\sin 30^\circ = -1/2$$

- (iv) This angle lies in the fourth quadrant. So $\sin 300^\circ$ should be negative.

$$\sin 300^\circ = \sin(-60^\circ) = -\sin 60^\circ = -\sqrt{3}/2$$

Illustration - 2 Determine the values of :

- (i) $\sin 180^\circ$ (ii) $\cos 180^\circ$ (iii) $\sin 270^\circ$ (iv) $\cos 270^\circ$

SOLUTION :

- (i) $\sin 180^\circ = \sin(180^\circ + 0^\circ) = -\sin 0^\circ = 0$ (ii) $\cos 180^\circ = \cos(180^\circ + 0^\circ) = -\cos 0^\circ = -1$

- (iii) $\sin 270^\circ = \sin(-90^\circ) = -\sin 90^\circ = -1$ (iv) $\cos 270^\circ = \cos(-90^\circ) = -\cos 90^\circ = 0$

Some Important Trigonometric Formulae :

1. (a) $\sin^2 \theta + \cos^2 \theta = 1$ (b) $1 + \tan^2 \theta = \sec^2 \theta$
 (c) $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$
2. Addition and subtraction formulae:
 (a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$ (b) $\sin(A - B) = \sin A \cos B - \cos A \sin B$
 (c) $\cos(A + B) = \cos A \cos B - \sin A \sin B$ (d) $\cos(A - B) = \cos A \cos B + \sin A \sin B$
 (e) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ (f) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$
3. Multiple formulae :
 (a) $\sin 2A = 2 \sin A \cos A$ (b) $\cos 2A = \cos^2 A - \sin^2 A$
 (c) $\cos 2A = 1 - 2 \sin^2 A = 2 \cos^2 A - 1$ (d) $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

Illustration - 3 Determine the values of

- (i) $\sin 75^\circ$ (ii) $\cos 75^\circ$

SOLUTION :

$$(i) \quad \sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}$$

$$(ii) \quad \cos 75^\circ = \cos(45^\circ + 30^\circ) = \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ = \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

Illustration - 4 Determine the values of

- (i) $\sin 22.5^\circ$ (ii) $\cos 22.5^\circ$

SOLUTION :

$$(i) \quad \text{Using } \cos 2\theta = 1 - 2 \sin^2 \theta \quad (ii) \quad \text{Using } \cos 2\theta = 2 \cos^2 \theta - 1$$

$$\sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}} \quad \cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}}$$

$$\therefore \sin 22.5^\circ = \sqrt{\frac{1 - \cos 45^\circ}{2}} = \sqrt{\frac{1}{2} - \frac{1}{2\sqrt{2}}} \quad \therefore \cos 22.5^\circ = \sqrt{\frac{1 + \cos 45^\circ}{2}} = \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{2}}}$$

IN-CHAPTER EXERCISE - A

Determine the values of :

1. $\cos 120^\circ$ 2. $\sin 150^\circ$ 3. $\cos 135^\circ$ 4. $\cos(-60^\circ)$
 5. $\sin 127^\circ$ 6. $\cos 143^\circ$ 7. $\tan 75^\circ$ 8. $\sin 15^\circ$

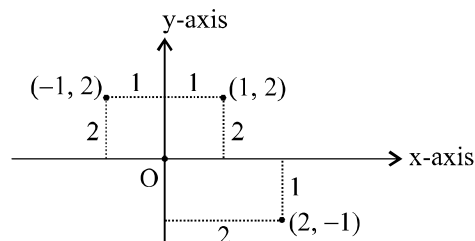
COORDINATE GEOMETRY

Section - 2

In Physics, we are often concerned with marking the location of an object or identifying the curve along which it moves. For example, you might have heard that planets revolve around elliptical paths around Sun and that a stone thrown near the surface of Earth moves in a parabolic path. For these, and many other other practical applications, an in depth knowledge of 'coordinate geometry' is required. We will first begin with some basic definitions and formulae.

Position of a point :

Position of any point is represented by a set of numbers (x, y) known as the cartesian coordinates of the point. Here x is the perpendicular distance of point from y -axis and y is the perpendicular distance from x -axis.

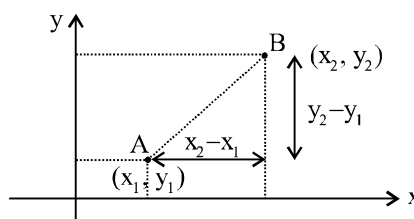


The point 'O' where the x -axis and y -axis intersect, is called the origin. The coordinates of the origin are $(0, 0)$.

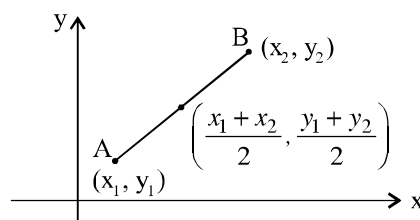
Distance between Two Points :

Formula for distance between two points is

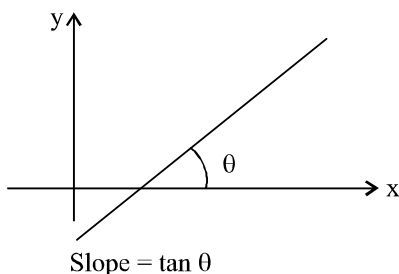
$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**Midpoint :**

Midpoint of a line joining $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

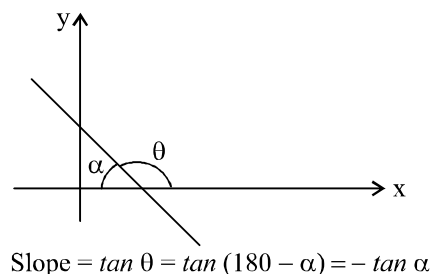
**Gradient (Slope) of a Line :**

Gradient of a straight line is a measure of steepness of the straight line. If a line makes an angle θ with +ve direction of x -axis, θ being taken in anticlockwise direction, then slope of the straight line is defined as $m = \tan \theta$



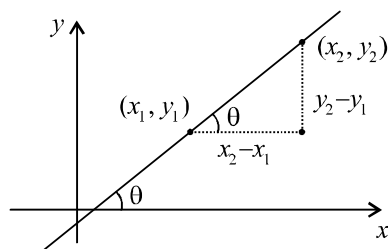
If θ is acute, slope is +ve

If θ is obtuse, slope is -ve

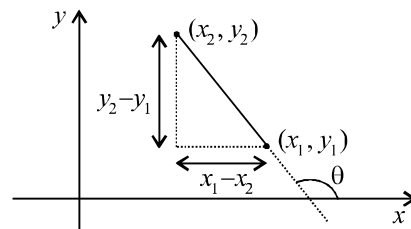


If (x_1, y_1) and (x_2, y_2) are two points on a straight line then slope of the straight line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{i.e.,} \quad m = \frac{\text{Rise in } y}{\text{Rise in } x}$$



$$m = \tan \theta = \frac{y_2 - y_1}{x_2 - x_1}$$



Straight line :

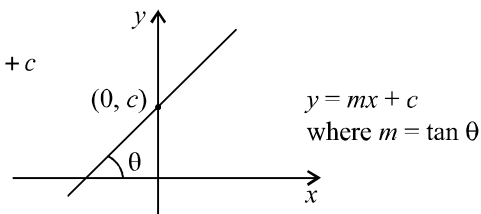
A straight line in a plane is defined as the set of points whose coordinates satisfy a linear equation $ax + by + c = 0$ where a , b and c are constants and (x, y) is a point on the straight line.

Equation of a straight line can be expressed in various other forms.

Equation of a straight line: Slope – Intercept form

Equation of a straight line with slope m and y -intercept c is given by $y = mx + c$

y -intercept of a line is the y coordinate of the point at which it crosses the y -axis.



Equation of a Straight Line : Two Intercept form

Equation of a straight line with x -intercept $= a$ and y -intercept $= b$ is given by :

$$\frac{x}{a} + \frac{y}{b} = 1$$

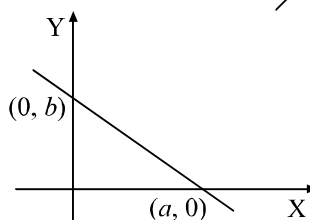


Illustration - 5

Write the equation of a straight line making intercepts of 4 units and 3 units on x and y -axis respectively.

Find the slope of straight line.

SOLUTION :

The line passes from the points $(4, 0)$ and $(0, 3)$.
Hence,

$$\text{slope} = \tan \theta = \frac{-3}{4}$$

Using $y = mx + c$, we can write the equation as:

$$y = -\frac{3}{4}x + 3 \Rightarrow 3x + 4y - 12 = 0$$

or by using two intercept form, $\frac{x}{4} + \frac{y}{3} = 1 \Rightarrow 3x + 4y - 12 = 0$

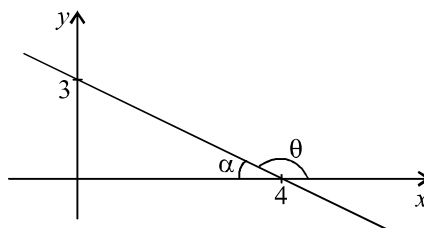


Illustration - 6 Write the equation of a straight line which makes an angle of 60° with the x -axis and cuts the y -axis at $y = -1$.

SOLUTION :

$$\text{Slope} = m = \tan 60^\circ = \sqrt{3}$$

$$y\text{-intercept} = c = -1$$

$$\text{Equation of the straight line : } y = \sqrt{3}x - 1$$

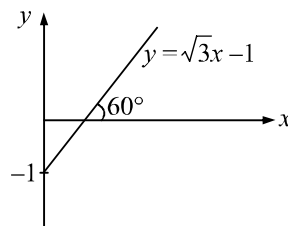


Illustration - 7 Write the equation of a line which cuts the y -axis at $y = 2$ and x -axis at $x = -3$.

SOLUTION :

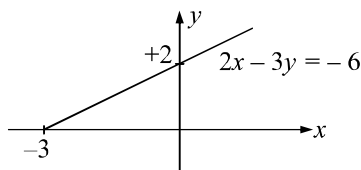
$$a = x \text{ intercept} = -3$$

$$b = y \text{ intercept} = +2$$

By two intercept form of a straight line :

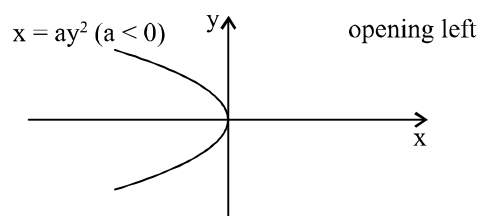
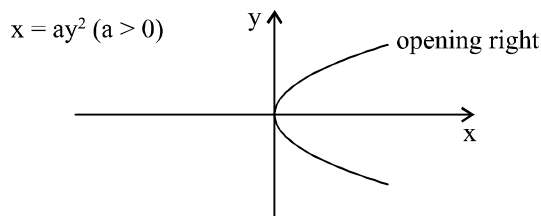
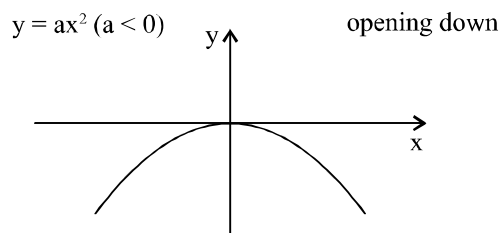
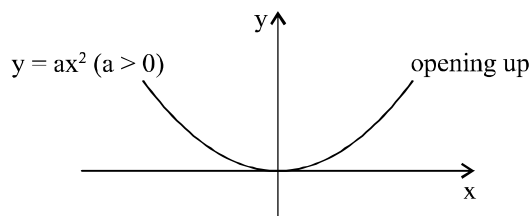
$$\frac{x}{-3} + \frac{y}{+2} = 1$$

$$\Rightarrow 2x - 3y = -6$$



Parabola :

In its simplest form a parabola is given by equation $y = ax^2$ or $x = ay^2$; where a is a non-zero constant. These parabolas pass through origin



In more general case, equations in the form $y = ax^2 + bx + c$ and $x = ay^2 + by + c$ to represent graph of a parabola.

$$y = ax^2 + bx + c$$

parabola opening up if $a > 0$

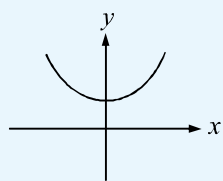
parabola opening down if $a < 0$

$$x = ay^2 + by + c$$

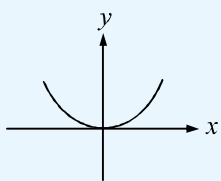
parabola opening right if $a > 0$

parabola opening left if $a < 0$

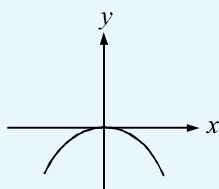
Illustration - 8 Which of the following graphs represents the given equation : $y = 2 - 5x^2$.



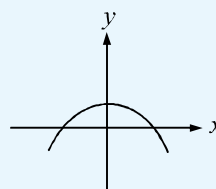
(A)



(B)



(C)



(D)

SOLUTION : (D)

(D) is the correct option as equation is of the form $y = ax^2 + bx + c$, where $a = -5 < 0$ and $y = 2$ when $x = 0$

⇒ Graph is a parabola opening downwards and cutting y-axis at $y = +2$

IN-CHAPTER EXERCISE - B

In the following problems plot the points and compute the slope of the line containing them. Graph the line.

1. $(2, 3), (1, 0)$

2. $(-2, 3), (2, 1)$

3. $(-3, -1), (2, -1)$

4. $(-1, 2), (-1, -2)$

5. $(\sqrt{2}, 3), (1, \sqrt{3})$

In the following problems graph the line passing through the point P and having slope m.

6. $P = (1, 2); m = 2$

7. $P = (2, 4); m = -3$

8. $P = (-1, 3); m = 0$

9. $P = (0, 3);$ slope undefined

In the following problems find a general equation for the line with the given properties.

10. Slope = 2; passing through $(-2, 3)$

11. Slope = $-2/3$; passing through $(1, -1)$

12. Passing through $(1, 3)$ and $(-1, 2)$

13. Slope = -3 ; y-intercept = 3

14. x-intercept = 2; y-intercept = -1

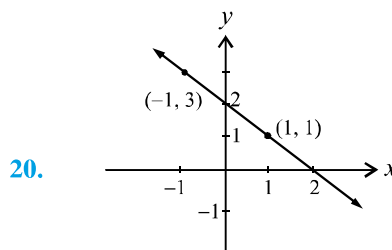
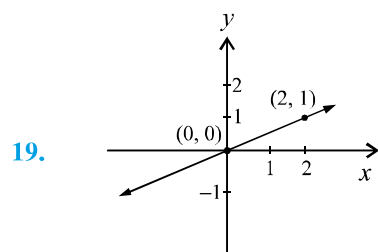
15. Slope undefined; passing through $(1, 4)$

16. Parallel to the line $y = 3x$; passing through $(-1, 2)$

17. Parallel to the line $2x - y + 2 = 0$; passing through $(0, 0)$

18. Parallel to the line $x = 5$; passing through $(4, 2)$

In the following problems find the general equation of each line.



LOGARITHMS

Section - 3

Logarithm is the reverse of power. It is just another way of writing an equation involving an exponent. If $a^k = c \Leftrightarrow \log_a c = k$ i.e. $\log_a c$ is 'to what power should I raise a to get c'. a is called base of the logarithm ($a > 0; a \neq 1$).

Example :

$$2^3 = 8 \Rightarrow \log_2 8 = 3$$

$$(25)^{\frac{1}{2}} = 5 \Rightarrow \log_{25} 5 = 0.5$$

$$\log_a (1) = 0 \text{ as } a^0 = 1$$

Common logarithm : Logarithm with base '10'. In general $\log x$ means common logarithm $\log x = \log_{10} x$

Natural logarithm :

In Physics, we mostly use exponents with base 'e'. So logarithm with base 'e' has been given a special name. $\log_e x = \ln x$

Properties of log :

It can be easily proved that following properties are true for 'log' (Proofs will be done in maths):

$$\log_a (xy) = \log_a (x) + \log_a (y)$$

$$\log_a \left(\frac{x}{y} \right) = \log_a x - \log_a y$$

$$\log_a x^p = p \cdot \log_a x$$

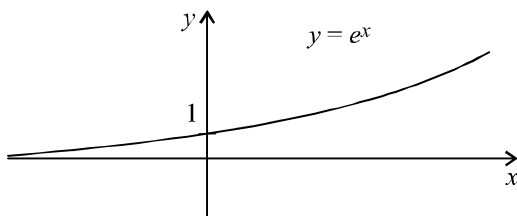
Base change formula:

$$\log_a x = \frac{\log_b x}{\log_b a} \text{ where } b \text{ is any number satisfying conditions of base.}$$

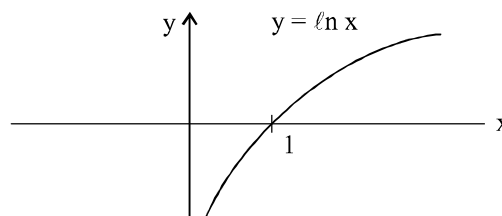
Example:

$$\log_2 5 = \frac{\log 5}{\log 2} = \frac{\ln 5}{\ln 2}$$

Graphs : It will be highly useful to understand and remember the following graphs:



Exponential



logarithm

Illustration - 9 If $\log_2 x = 5$, find x .**SOLUTION :**

$$\log_2 x = 5 \Rightarrow x = 2^5 = 32$$

Illustration - 10 Solve for x : $\log x = (\log 27)/3$ **SOLUTION :**

$$\log x = \frac{\log(27)}{3} = \frac{\log(3^3)}{3} = \log(3^3)^{1/3}$$

$$\Rightarrow \log x = \log 3 \Rightarrow x = 3$$

Illustration - 11 Find the value of (Given $\ln 2 = 0.7$)

- (A) $\ln 8$ (B) $\ln \sqrt{2}$ (C) $\ln(1/64)$

SOLUTION :

(A) $\ln 8 = \ln(2^3) = 3(\ln 2) = 3(0.7) = 2.1$ (B) $\ln(\sqrt{2}) = \frac{1}{2} \ln(2) = \frac{1}{2}(0.7) = 0.35$

(C) $\ln\left(\frac{1}{64}\right) = \ln(2^{-6}) = -6(0.7) = -4.2$

Illustration - 12 If $\log_{10} 2 = 0.3010$, the value of $\log_{10} 80$ is:

- (A) 1.6020 (B) 1.9030 (C) 3.9030 (D) None of these

SOLUTION : (B)

$$\begin{aligned} \log_{10} 80 &= \log_{10}(8 \times 10) \\ &= \log_{10} 8 + \log_{10} 10 = \log_{10}(2^3) + 1 \\ &= 3 \log_{10} 2 + 1 = 3(0.3010) + 1 = 1.9030. \end{aligned}$$

IN-CHAPTER EXERCISE - C

1. Calculate the value of $(\log_2 100) - (\log_2 10)$. Given that $\log_{10} 2 = 0.3$.
2. Using the properties of logs, write in expanded form: $\log(2ab/c^3)$
3. Express as a single logarithm: $(2(\ln a - \ln b) - 5 \ln c)/2$.

Choose the correct alternative. Only One Choice is Correct :

4. The value of $\log_2 16$ is:
 (A) 1/8 (B) 4 (C) 8 (D) 16
5. If $a^x = b^y$, then:
 (A) $\log \frac{a}{b} = \frac{x}{y}$ (B) $\frac{\log a}{\log b} = \frac{x}{y}$ (C) $\frac{\log a}{\log b} = \frac{y}{x}$ (D) None of these
6. If $\log_{10} 7 = a$, then $\log_{10} \left(\frac{1}{70} \right)$ is equal to:
 (A) $-(1+a)$ (B) $(1+a)^{-1}$ (C) $\frac{a}{10}$ (D) $\frac{1}{10a}$
7. If $\log_{10} 2 = 0.3010$, then $\log_2 10$ is equal to:
 (A) $\frac{699}{301}$ (B) $\frac{1000}{301}$ (C) 0.3010 (D) 0.6990
8. Which of the following statements is not correct?
 (A) $\log_{10} 10 = 1$ (B) $\log(2+3) = \log(2 \times 3)$
 (C) $\log_{10} 1 = 0$ (D) $\log(1+2+3) = \log 1 + \log 2 + \log 3$
9. If $\log 2 = 0.3010$ and $\log 3 = 0.4771$, the value of $\log_5 512$ is:
 (A) 2.870 (B) 2.967 (C) 3.876 (D) 3.912
10. If $\log \frac{a}{b} + \log \frac{b}{a} = \log(a+b)$, then:
 (A) $a+b=1$ (B) $a-b=1$ (C) $a=b$ (D) $a^2-b^2=1$

DIFFERENTIATION

Section - 4

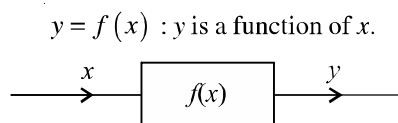
Function :

Functions are a convenient way to express relationship between two variable quantities. For example, let us imagine a spherical rubber balloon into which air is being pumped. The radius of the balloon (r) and its volume (V) will change (increase) with time. In mathematics, we say that r is a function of time. This is written mathematically as, $r = f(t)$: r is a function of time t .

Similarly volume depends on time t

$$V = g(t) : V \text{ is a function of time } t$$

Different letters f and g are used because they represent different mathematical relations. In general, if value of a variable quantity y depends on the value of some other variable quantity x , through some relation, such that there is only one value of y for each value of x then y is called a function of x .



We can imagine function as a box in which a value of x is entered and it returns y by performing some defined operation on it.

Example :

1. $y = x^2$

2. $y = \sin x, y = \cos x$ (Trigonometric)

3. $y = \log_{10} x, y = \log_e x$

4. $y = 2^x, y = e^x$

Illustration - 13 A particle starts from rest and moves with a uniform acceleration of 2 m/s^2 .

(A) Express its velocity and displacement as a function of time (t).

(B) Find velocity and displacement at $t = 3 \text{ s}$.

SOLUTION :

(A) Velocity $v = u + at = 0 + 2t$

$$v = f(t) = 2t$$

$$S = ut + \frac{1}{2}at^2 = 0 \times t + \frac{1}{2} \times 2t^2 = t^2$$

(B) at $t = 3 \text{ s}$, $v = 2 \times 3 = 6 \text{ m/s}$

$$S = (3)^2 = 9 \text{ m}$$

Independent and dependent variables :

In the function $y = f(x)$, x is known as the independent variable and y (whose value depends on x) is known as dependent variable.

Differentiation (Derivative) of a function :

If change in x is smaller than any possible number, i.e. $\Delta x \rightarrow 0$ (but not equal to zero) then such a change is called an infinitesimal change. An infinitesimal change in x -coordinate is denoted by dx , in mass by dm , in time by dt etc. The process of calculating the ratio of two infinitesimal quantities may be called differentiation.

For example, the ratio $\frac{dy}{dx}$ is called the derivative of y with respect to x . It is also called the 'rate of change of y with respect

to x '. The derivative can formally be defined as, $\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$

- Note :**
1. The limit $\Delta x \rightarrow 0$ does not imply that Δx is equal to zero. It means Δx has become so small that it is almost approaching or tending to zero.
 2. We have defined differentiation in a very rudimentary manner here. This will suffice as far as 11th, 12th physics is concerned. In mathematics, you will study differentiation in a much greater depth at a later stage.

In the following illustration, we will try to understand how a derivative is calculated.

Illustration - 14 Determine the derivative of $y = x^2$ & find its value at $x = 1$.

SOLUTION :

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{x^2 + (\Delta x)^2 + 2x\Delta x - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \Delta x + 2x = 2x \end{aligned}$$

Value of this derivative at $x = 1$ is written as $\left. \frac{dy}{dx} \right|_{x=1} = 2$

Differentiation formulae for common functions :

(i) **Algebraic function:** $\frac{d}{dx}(x^n) = nx^{n-1}$

For example you can verify the following derivatives :

$$\frac{d(x^2)}{dx} = 2x, \quad \frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}, \quad \frac{d\left(\frac{1}{x}\right)}{dx} = -\frac{1}{x^2}, \quad \frac{d\left(\frac{1}{\sqrt{x}}\right)}{dx} = \frac{-1}{2x\sqrt{x}}$$

(ii) **Trigonometric functions :**

$$\begin{aligned} \frac{d(\sin x)}{dx} &= \cos x & \frac{d(\cos x)}{dx} &= -\sin x & \frac{d(\tan x)}{dx} &= \sec^2 x \\ \frac{d(\sec x)}{dx} &= \sec x \tan x & \frac{d(\operatorname{cosec} x)}{dx} &= -\operatorname{cosec} x \cot x & \frac{d(\cot x)}{dx} &= -\operatorname{cosec}^2 x \end{aligned}$$

(iii) **Constant function :** $\frac{d}{dx}(\text{constant}) = \text{Zero}$

(iv) Logarithmic Function: $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(v) Exponential Function: $\frac{d}{dx}(e^x) = e^x$

Basic Rules for Differentiation:

Addition/Subtraction Rule : If $y = u + v - w$, where u , v and w are function of x , then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$

Product Rule: If $y = uv$, where u and v are functions of x , then $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

Quotient Rule: If $y = \frac{u}{v}$ where u and v are functions of x , then $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

Note : If $y = cv$, where c is a constant and v is a function of x , then the constant can be taken out of the derivative function.

$$\frac{dy}{dx} = \frac{d(cv)}{dx} = c \frac{dv}{dx}$$

Illustration - 15 Find the derivatives of following functions:

(A) $\sqrt{2x}$

(B) $x^2 + \sin x$

(C) $\frac{\tan x}{2} + 3 \sec x - \log_e x$

(D) $(x^2 + 1)(x + 2)$

(E) $\frac{\log_e x}{x}$

SOLUTION :

(A) $\frac{d(\sqrt{2x})}{dx} = \frac{\sqrt{2} d\sqrt{x}}{dx} = \sqrt{2} \times \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{2x}}$

(B) $\frac{d(x^2 + \sin x)}{dx} = \frac{dx^2}{dx} + \frac{d \sin x}{dx} = 2x + \cos x$

(C) $\frac{d\left(\frac{\tan x}{2} + 3 \sec x - \log_e x\right)}{dx} = \frac{1}{2} \frac{d}{dx}(\tan x) + 3 \frac{d}{dx}(\sec x) - \frac{d}{dx}(\log_e x)$
 $= \frac{\sec^2 x}{2} + 3 \sec x \tan x - \frac{1}{x}$

(D) $\frac{d}{dx}(x^2 + 1)(x + 2) = (x^2 + 1) \frac{d}{dx}(x + 2) + (x + 2) \frac{d}{dx}(x^2 + 1)$
 $= (x^2 + 1)(1) + (x + 2)(2x)$
 $= 3x^2 + 4x + 1$

(E) $\frac{d}{dx}\left(\frac{\log_e x}{x}\right) = \frac{x \frac{d}{dx}(\log_e x) - \log_e x \frac{dx}{dx}}{x^2} = \frac{x \times \frac{1}{x} - \log_e x}{x^2} = \frac{1 - \log_e x}{x^2}$

Derivative of Linear Extension of a Function :

If $\frac{d}{dx} f(x) = g(x)$ then $\frac{d}{dx} f(ax+b) = a \times g(ax+b)$

e.g., $\frac{d}{dx} x^5 = 5x^4 \Rightarrow \frac{d}{dx} (3x+4)^5 = 3[5(3x+4)^4] = 15(3x+4)^4$

Illustration - 16 Find the derivative of following functions.

(A) $(2x+3)^5$ (B) $\tan(5x-1)$

SOLUTION :

(A) $y = (2x+3)^5 \Rightarrow \frac{dy}{dx} = 2[5(2x+3)^4] = 10(2x+3)^4$

(B) $\frac{d}{dx} \tan x = \sec^2 x \Rightarrow \frac{d}{dx} \tan(5x-1) = 5[\sec^2(5x-1)]$

IN-CHAPTER EXERCISE-D

Differentiate each function by applying the basic rules for differentiation.

1. $x^5 - 3x^3 + 1$

2. $\frac{x^{10}}{2} + \frac{x^5}{5} + 6$

3. $t^8 - 2t^7 + 3t + 1$

4. $\frac{3}{x^2} + \frac{4}{x}$

5. $3x^{-2} - 7x^{-1} + 6$

6. $\frac{2}{5x} - \frac{\sqrt{2}}{3x^2}$

7. $x^2(3x^3 - 1)$

8. $(x^2 + 3x)(x^3 - 9x)$

9. $(x^3 - 8)\left(\frac{2}{x} - 1\right)$

10. $\left(\frac{1}{x^2} + 3\right)\left(\frac{2}{x^3} + x\right)$

11. π

12. $(x-1)^2$

13. $1/\sqrt{x}$

14. $1/x\sqrt{x}$

15. $ax^2 + bx + c$

16. $\sin x + \cos x$

17. $\sin^2 x$

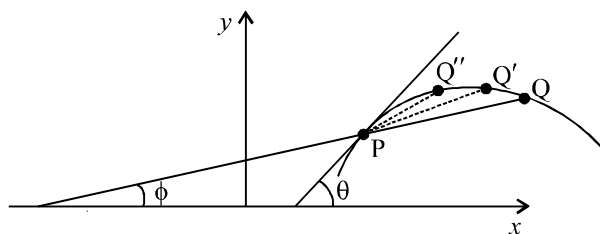
18. $1 + \tan x$

19. $\ln x - e^x$

20. e^{ax+b}

Geometrical Significance of Derivative :

Consider the graph of function $y = f(x)$ in which two points $P(x, y)$ & $Q(x + \Delta x, y + \Delta y)$ have been marked.



Slope of chord joining PQ is given by $\tan \phi = \frac{\Delta y}{\Delta x}$

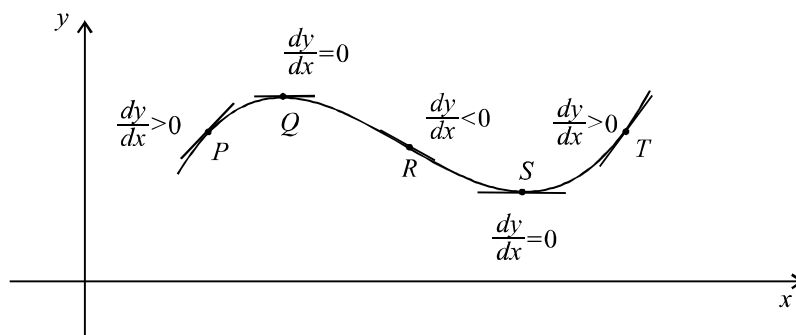
Now, as point Q is brought closer to P , Δx becomes progressively smaller. The slope of chord changes as we move from Q towards P ($PQ, PQ', PQ'' \dots$). As point Q approaches P , Δx approaches zero, but at the same time the slope of the dotted line approaches that of the tangent to the curve at P .

$$\text{Slope of tangent at } P = \tan \theta = \lim_{\Delta x \rightarrow 0} \tan \phi = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

Hence, the derivative of a function $y = f(x)$ at a point gives slope of the tangent at that point.

Also, if $\frac{dy}{dx} > 0$ at a point then tangent at that point makes an acute angle with x -axis.

If $\frac{dy}{dx} < 0$ at a point then tangent at that point makes an obtuse angle with x -axis.



It can also be seen from the graph that the value of y increases with x when $\frac{dy}{dx}$ is +ve & value of y decreases with x when $\frac{dy}{dx}$ is -ve.

In physics, derivative of a function represents the rate of change of one quantity with respect to other.

For example, rate of change of position of a particle with respect to time $\frac{dx}{dt}$ is called velocity of the particle.

$$v = \frac{dx}{dt}$$

Similarly rate of change of velocity w.r.t time, $\frac{dv}{dt}$ gives acceleration of the particle $a = \frac{dv}{dt} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d^2x}{dt^2}$.

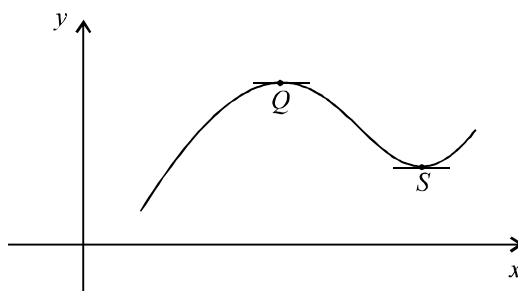
Acceleration of a particle 'a' can also be written as $a = \frac{dv}{dt} = \frac{dv}{dx} \left(\frac{dx}{dt} \right) = v \frac{dv}{dx}$

In terms of slope

Slope of a displacement time graph gives velocity at a point & slope of a velocity time graph gives acceleration at that point.

Maxima & Minima :

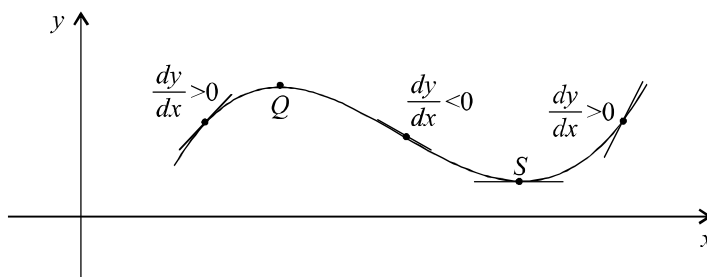
Maximum & minimum value of a function can be determined using calculus in the following manner. Consider the graph of a function $y = f(x)$:



It can be seen from graph that at Q (maximum) or S (minimum), derivative $\frac{dy}{dx} = 0$. Hence, the condition for a local maximum or minimum is that the derivative of the function is zero at that point.

The next question is that how do we conclude if the point is a maximum or minimum. In most problems of Physics, this can be ascertained easily by looking at the conditions of the problem. But still, if required, this can also be determined mathematically as follows.

To ascertain mathematically whether a point where $\frac{dy}{dx} = 0$ is a maximum or minimum, we need to check how $\frac{dy}{dx}$ changes around Q or S .



Around Q , $\frac{dy}{dx}$ changes from positive $\rightarrow$ zero $\rightarrow$ negative

Around S, $\frac{dy}{dx}$ changes from negative $\rightarrow$ zero $\rightarrow$ positive

Mathematically, double derivative $\frac{d^2y}{dx^2}$ represents how $\frac{dy}{dx}$ changes. i.e. $\frac{d^2y}{dx^2}$ is derivative of $\frac{dy}{dx}$

OR
$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

Therefore, the above condition can be mathematically put as:

at Q; $\frac{d^2y}{dx^2} < 0$ at S; $\frac{d^2y}{dx^2} > 0$ i.e.; If the double derivative at the point is negative, it is a maximum. If the double derivative is positive, it is a minimum. The complete result can be summarised as follows.

Maxima/Minima Summary :

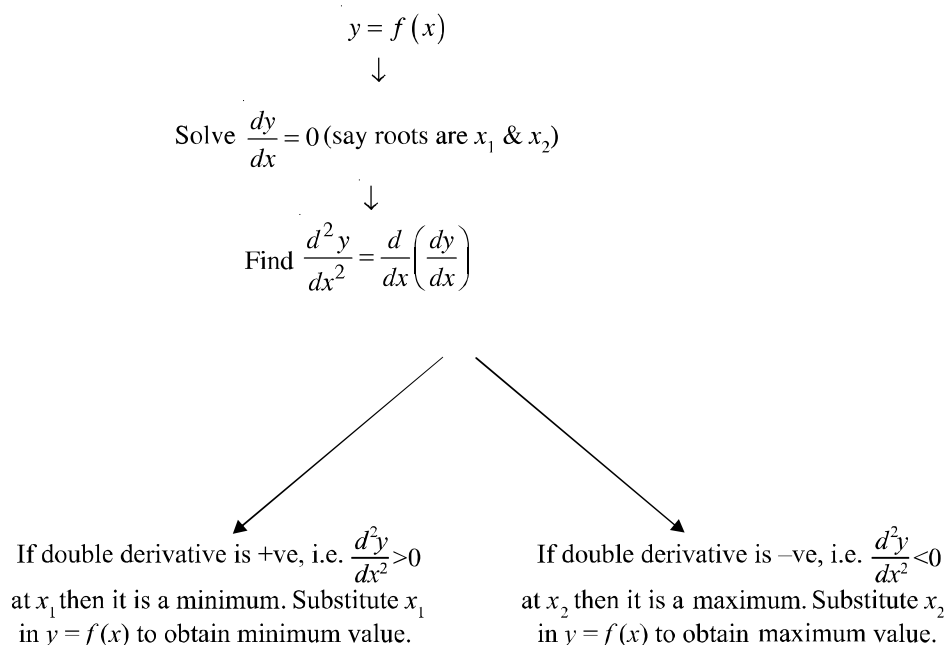


Illustration - 17 Find the slopes of tangents drawn on curve $y = x^2$ at $x = 0$ & $x = 1/2$. Also find the angle made by these tangents with x-axis.

SOLUTION :

Slope of tangent $= \frac{dy}{dx} = \frac{d(x^2)}{dx} = 2x$

at $x = 0$; $\left. \frac{dy}{dx} \right|_{x=0} = 2 \times 0 = 0 \Rightarrow \theta = 0^\circ$

at $x = \frac{1}{2}$; $\left. \frac{dy}{dx} \right|_{x=\frac{1}{2}} = 2 \times \frac{1}{2} = 1 \Rightarrow \theta = 45^\circ$

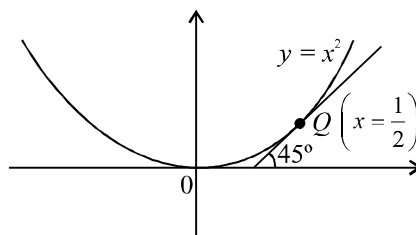


Illustration - 18Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ if

(A)

$y = 4\sin(2x + 5)$

(B)

$y = x \ln x$

SOLUTION :

$$(A) \quad \frac{dy}{dx} = \frac{d[4\sin(2x+5)]}{dx} = 8\cos(2x+5)$$

$$\frac{d^2y}{dx^2} = \frac{d[8\cos(2x+5)]}{dx} = -16\sin(2x+5)$$

$$(B) \quad \frac{dy}{dx} = \frac{d(x \ln x)}{dx} = \ln x + \frac{x}{x} = 1 + \ln x$$

$$\frac{d^2y}{dx^2} = \frac{d(1 + \ln x)}{dx} = \frac{1}{x}$$

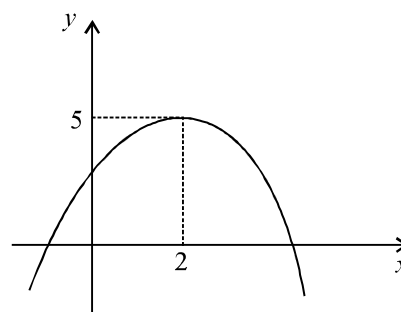
Illustration - 19If $y = 4x - x^2 + 1$, find maximum value of y .**SOLUTION :**

$$\frac{dy}{dx} = 4 - 2x = 0$$

$$\Rightarrow x = 2$$

$$\frac{d^2y}{dx^2} = -2 < 0, \text{ So } x = 2 \text{ is a position of maxima}$$

$$y_{\max} = 4(2) - (2)^2 + 1 = 8 - 4 + 1 = 5$$

**Illustration - 20**

A small body of mass 1 kg is to be broken into two parts such that gravitational force between them is maximum. Find the mass of the two broken parts.

SOLUTION :Let the mass be m and $(1 - m)$

$$F = \frac{Gm(1-m)}{r^2}$$

 F depends on m for a given r . For maximum force,

$$\frac{dF}{dm} = 0$$

$$\frac{d}{dm} \left[\frac{Gm(1-m)}{r^2} \right] = 0$$

$$\frac{G}{r^2} \frac{d}{dm} (m - m^2) = 0$$

$$\frac{G}{r^2} (1 - 2m) = 0 \Rightarrow m = \frac{1}{2}$$

$$\frac{d^2F}{dm^2} = \frac{G}{r^2} (-2) = -ve$$

 $\therefore$ For $m = \frac{1}{2}$, we get maxima.

So the mass of two parts should be 0.5 kg each for getting maximum force

INTEGRATION

Section - 5

Integration can be said to be the reverse process of differentiation.

The symbol used for integration is: $\int () dx$. The function that is to be integrated is written inside the bracket. For example,

‘Integral of $\sin x$ ’ is written as $\int (\sin x) dx$.

Most of the formulae for integration can be derived by using the fact that integration is the reverse process of differentiation.

For example, we know that $\frac{d(\sin x)}{dx} = \cos x$

Then, $\int (\cos x) dx$ will be a function, whose differentiation will give back ‘ $\cos x$ ’. From our knowledge of differentiation, it is easily seen that such a function is ‘ $\sin x$ ’. So, $\int \cos x dx = \sin x$

Also, since differentiation of a constant is zero, a more general way of writing the answer is: $\int \cos x dx = \sin x + c$

In Physics, value of this constant can be derived by using the conditions given in the problem.

For those of you who love mathematics, the concept explained above with the help of a simple example can be written in

general mathematical language as, If $\frac{d(f(x))}{dx} = g(x)$ then, $d(f(x)) = g(x) dx$

$$\Rightarrow f(x) = \int g(x) dx, \quad f(x) \text{ is the integral of } g(x)$$

Basic Integration Formulae :

$$1. \quad \int x^n dx = \frac{x^{n+1}}{n+1} + c \quad (n \neq -1)$$

$$\text{Examples. } \int dx = x, \quad \int x dx = \frac{x^2}{2}, \quad \int \sqrt{x} dx = \frac{x^{3/2}}{3/2}, \quad \int \frac{dx}{x^2} = \frac{-1}{x}$$

$$2. \quad \int \frac{dx}{x} = \ln x + c$$

$$3. \quad \int e^x dx = e^x + c$$

$$4. \quad \int \sin x dx = -\cos x + c$$

$$5. \quad \int \cos x dx = \sin x + c$$

$$6. \quad \int \sec^2 x dx = \tan x + c$$

$$7. \quad \int \cos ec^2 x dx = -\cot x + c$$

$$8. \quad \int \sec x \tan x dx = \sec x + c$$

$$9. \quad \int \co sec x \cot x dx = -\cos ec x + c$$

Basic Rules for Integration :

$$1. \quad \int c f(x) dx = c \int f(x) dx \quad \text{where } c \text{ is a const.} \quad 2. \quad \int (f(x) \pm g(x)) dx = \int f(x) dx \pm \int g(x) dx$$

$$3. \quad \text{If } \int f(x) dx = g(x) + c \quad \text{then } \int f(ax+b) dx = \frac{g(ax+b)}{a} + c$$

Illustration - 21 Evaluate the following integrals

(A) $\int (x+x^2) dx$ (B) $\int \left(e^x + \frac{1}{x}\right) dx$ (C) $\int \sin(2x) dx$ (D) $\int \frac{dx}{3x+2}$ (E) $\int (-3x+2)^{-5} dx$

SOLUTION :

(A) $\int (x+x^2) dx = \int x dx + \int x^2 dx = \frac{x^2}{2} + \frac{x^3}{3} + c$ (B) $\int \left(e^x + \frac{1}{x}\right) dx = \int e^x dx + \int \frac{dx}{x} = e^x + \ln x + c$

(C) $\int \sin(2x) dx = \frac{-\cos(2x)}{2} + c$ (D) $\int \frac{dx}{3x+2} = \frac{\ln(3x+2)}{3} + c$

(E) $\int (2-3x)^{-5} dx = \frac{(2-3x)^{-5+1}}{(-5+1)} \times \frac{1}{-3} + c = \frac{(2-3x)^{-4}}{-4} \times \frac{1}{-3} + c = \frac{1}{12(2-3x)^4} + c$

Definite Integration :

When integration is performed between definite limits, it is called definite integration. To calculate a definite integral, we first find its indefinite integral and then substitute upper & lower limits & subtract.

If $\int f(x) dx = g(x) + c$ then $\int_a^b f(x) dx = g(b) - g(a)$

Illustration - 22 Find the value of following integrals.

(A) $\int_2^4 \frac{dt}{t}$ (B) $\int_0^{\pi/4} \cos(3x) dx$ (C) $\int_1^2 \left(e^x - \frac{1}{x^2}\right) dx$ (D) $\int_0^1 (2x+3)^4 dx$

SOLUTION :

(A) $\int_2^4 \frac{dt}{t} = [\ln(t)]_2^4 = \ln(4) - \ln(2) = \ln\left(\frac{4}{2}\right) = \ln(2)$

(B) $\int_0^{\pi/4} \cos(3x) dx = \left[\frac{\sin 3x}{3}\right]_0^{\pi/4} = \frac{1}{3} \left[\sin \frac{3\pi}{4} - \sin 0\right] = \frac{1}{3\sqrt{2}}$

(C) $\int_1^2 \left(e^x - \frac{1}{x^2}\right) dx = \int_1^2 e^x dx - \int_1^2 \frac{dx}{x^2} = [e^x]_1^2 - \left[-\frac{1}{x}\right]_1^2 = (e^2 - e^1) + \left(\frac{1}{2} - 1\right) = e^2 - e - \frac{1}{2}$

(D) $\int_0^1 (2x+3)^4 dx = \left[\frac{(2x+3)^5}{10}\right]_0^1 = \frac{1}{10} [5^5 - 3^5] = \frac{2882}{10}$

IN-CHAPTER EXERCISE - E

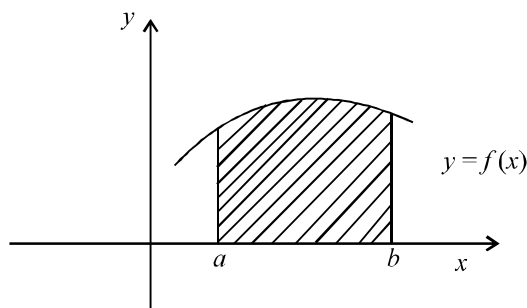
Integrate each of the following functions with respect to x .

1. (i) $\int x^4 dx$ (ii) $\int x^{5/4} dx$ (iii) $\int 1/x^5 dx$ (iv) $\int 1/x^{3/2} dx$ (v) $\int \frac{1}{\sqrt[3]{x^2}} dx$
2. $\int (3x\sqrt{x} + 4\sqrt{x} + 5) dx$ 3. $\int (\sqrt{x}(ax^2 + bx + c)) dx$ 4. $\int \sqrt{x} \left(x^3 - \frac{2}{x} \right) dx$ 5. $\int \frac{1}{\sqrt{x}} \left(1 + \frac{1}{x} \right) dx$
6. $\int \frac{x^5 + x^{-2} + 2}{x^2} dx$ 7. $\int (3x^2 - 4x - 5) dx$ 8. $\int \frac{x^3 - 1}{x - 1} dx$ 9. $\int \left(x^2 + 3x + \frac{1}{x^2} \right) dx$
10. $\int (2 \sin x + 3 \cos x) dx$ 11. $\int 2 \sin 35x dx$ 12. $\int 3 \cos(8x - 1) dx$ 13. $\int \sec x (\tan x + \sec x) dx$

Evaluate the following definite integrals

14. $\int_{-2}^0 (2x + 5) dx$ 15. $\int_0^4 \left(3x - \frac{x^3}{4} \right) dx$ 16. $\int_0^1 (x^2 + \sqrt{x}) dx$ 17. $\int_1^{32} x^{-6/5} dx$
18. $\int_0^{\pi} \sin x dx$ 19. $\int_{\pi/2}^0 \frac{1 + \cos 2t}{2} dt$ 20. $\int_1^{-1} (r + 1)^2 dr$

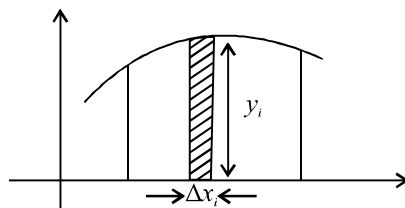
Definite Integral and Area



Area under the curve $y = f(x)$ and above x -axis between $x = a$ & $x = b$ [The area shaded in figure] is given by

$$\text{Area} = \int_a^b y dx$$

To find the area we divide entire area into small rectangles with height y_i & breadth Δx_i ($i = 1, 2, \dots, n$) value of y_i can be assumed constant over the interval & region can be treated as a rectangle if $\Delta x_i \rightarrow 0$ ($i = 1, 2, \dots, n$)



$$\text{Area} = y_1 \Delta x_1 + y_2 \Delta x_2 + \dots + y_n \Delta x_n = \sum_{i=1}^n y_i \Delta x_i$$

As $\Delta x_i \rightarrow 0$ & $n \rightarrow \infty$

$$\text{Area} = \int_a^b y dx$$

Area above x -axis is +ve & below x -axis is negative.

Illustration - 23 Find the area enclosed by $y = x^2$ and x -axis from $x = 0$ to $x = 2$.

SOLUTION :

$$\text{Area} = \int_0^2 y dx = \int_0^2 x^2 dx = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$$

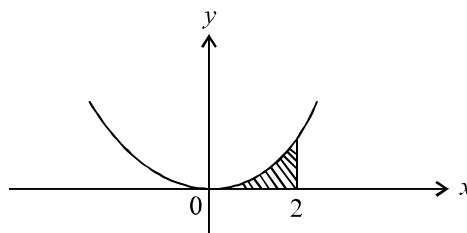
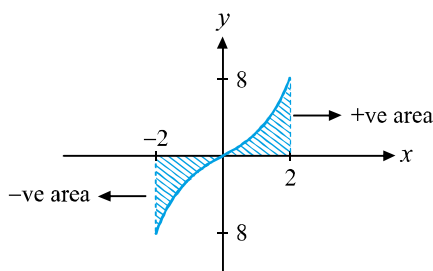


Illustration - 24 Find area under the graph $y = x^3$ between $x = -2$ and $x = 2$.

SOLUTION :

$$\begin{aligned} \text{Area} &= \int_{-2}^2 x^3 dx = \left[\frac{x^4}{4} \right]_{-2}^2 \\ &= \frac{1}{4} [2^4 - (-2)^4] = 0 \end{aligned}$$



MISCELLANEOUS EXERCISE

Choose the correct options for each of the following questions. Questions marked with * may have more than one correct options.

1. The equation of the straight line passing through the point (4, 3) and making intercepts on the coordinate axes, whose sum is -1, is:

(A) $\frac{x}{2} + \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$

(B) $\frac{x}{2} - \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$

(C) $\frac{x}{2} + \frac{y}{3} = 1$ and $\frac{x}{2} + \frac{y}{1} = 1$

(D) $\frac{x}{2} - \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$

2. A straight line through the point A(3, 4) is such that its intercept between the axes is bisected at A. Its equation is:

(A) $x + y = 7$ (B) $3x - 4y + 7 = 0$

(C) $4x + 3y = 24$ (D) $3x + 4y = 25$

3. If $\log_x y = 100$ and $\log_2 x = 10$, then the value of y is:

(A) 2^{10} (B) 2^{100}

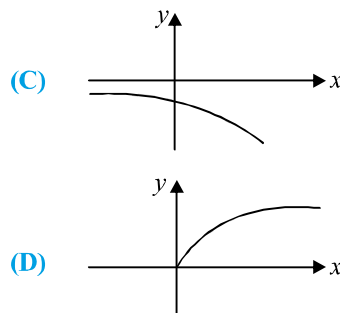
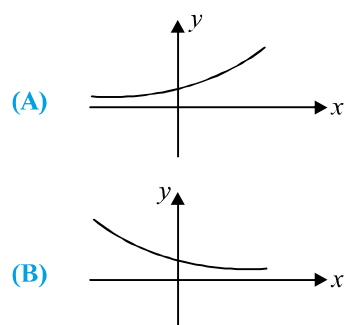
(C) 2^{1000} (D) 2^{10000}

4. If $\log_x \left(\frac{9}{16} \right) = -\frac{1}{2}$, then x is equal to:

(A) $-\frac{3}{4}$ (B) $\frac{3}{4}$

(C) $\frac{81}{256}$ (D) $\frac{256}{81}$

5. Choose the correct graph for $y = e^{-x}$



6. The rate of change of displacement with respect to time is known as velocity. If displacement of a particle is $s = 3t^2 + 4t + 5$, find velocity as function of time.

(A) $3t^2 + 4t + 5$ (B) $6t + 4$

(C) $3t^2 + 4$ (D) $10t + 5$

7. The rate of change of velocity with respect to time is acceleration of a particle. The velocity of a particle is $v = 3 \sin t$. Find acceleration of the particle as function of time.

(A) $3 \sin t$ (B) $3 \cos t$

(C) $-3 \cos t$ (D) $-3 \sin t$

8. Equation of tangent drawn to curve $y = x^2$ at $x = 1$ is

(A) $y = 2x$ (B) $y = 2x - 1$

(C) $y = x$ (D) $y = \frac{x}{2} + 1$

9. Evaluate $\int_{20}^{40} \frac{dx}{10 - x}$

(A) $\ln(3)$ (B) $-\ln(3)$

(C) $\ln(2)$ (D) $-\log(3)$

10. If $\frac{dy}{dx} = 2x + 1$, then find the value of y in terms of x if $y = 3$ when $x = 1$:

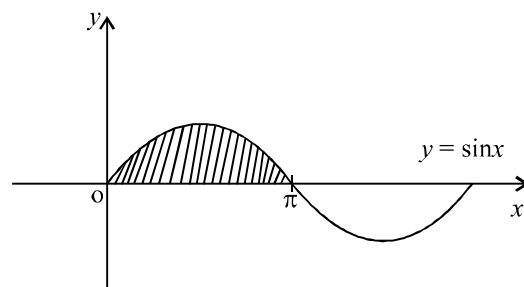
(A) $y = x^2 + x + 1$ (B) $y = x^2 - x + 3$

(C) $y = 2x^2 - x + 2$ (D) $y = x^2 + x - 1$

11. Force between two particles varies as $F = \frac{K}{r^2}$, where k is a constant and r is the separation between the particles. Potential energy (u) of the system is related to force as $F = -\frac{du}{dr}$, then u is (here u_0 is an arbitrary constant).

- (A) $-\frac{k}{r} + u_0$ (B) $\frac{k}{r} + u_0$
(C) $\frac{k}{r^2} + u_0$ (D) $-\frac{k}{r^2} + u_0$

12. Shaded area is equal to



- (A) 1 unit (B) 2 units
(C) 0.5 units (D) 1.5 units

SOLUTIONS TO IN-CHAPTER EXERCISES

IN-CHAPTER EXERCISE-A

- $\cos 120^\circ = \cos (180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$
- $\sin 150^\circ = \sin (180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}$
- $\cos 135^\circ = \cos (180^\circ - 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}}$
- $\cos (-60^\circ) = \cos 60^\circ = \frac{1}{2}$
- $\sin 127^\circ = \sin (180^\circ - 53^\circ) = \sin 53^\circ = \frac{4}{5}$
- $\cos 143^\circ = \cos (180^\circ - 37^\circ) = -\cos 37^\circ = -\frac{4}{5}$

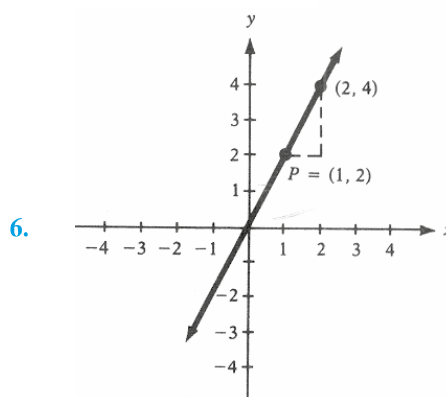
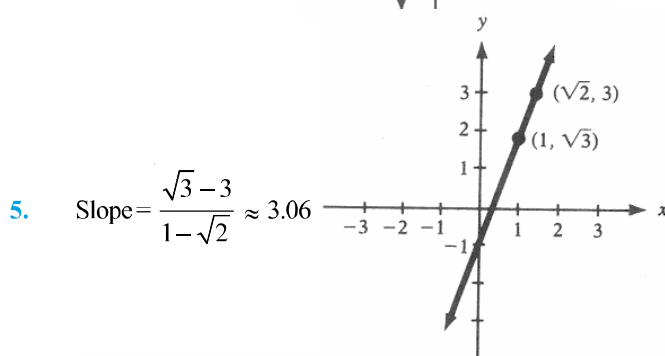
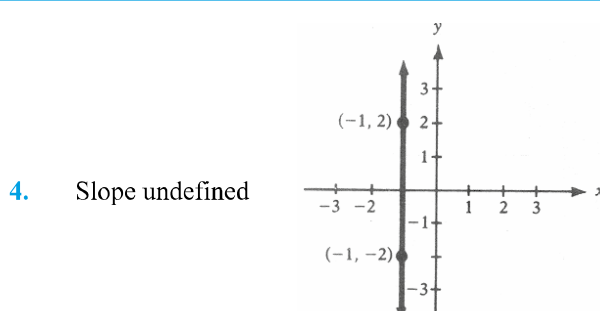
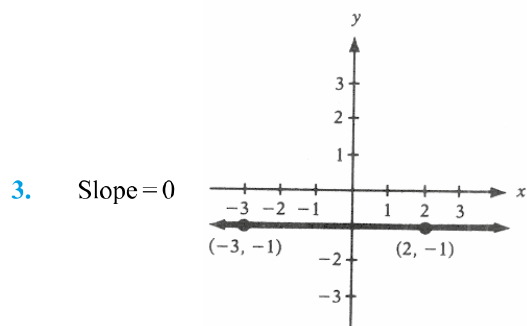
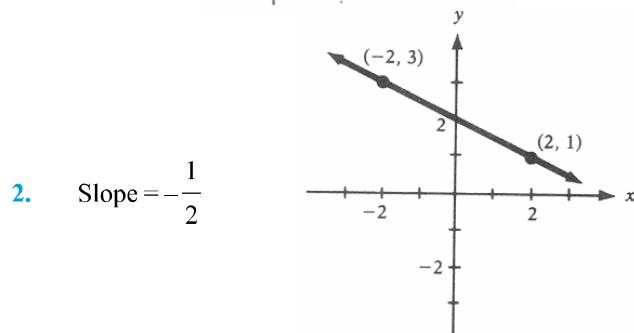
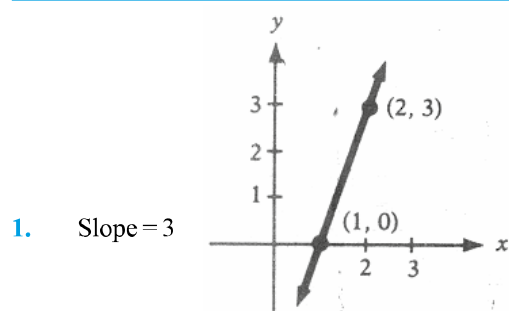
- $\tan 75^\circ = \tan (45^\circ + 30^\circ)$

$$= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$$
- $\sin 15^\circ = \sin (45^\circ - 30^\circ)$

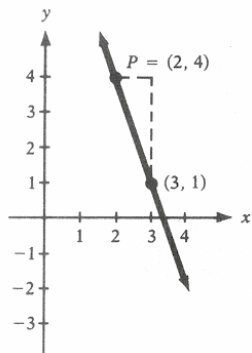
$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

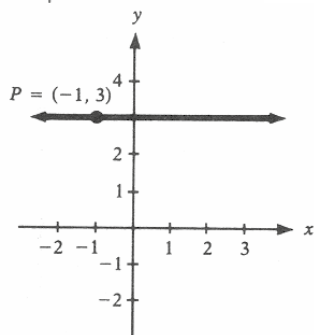
IN-CHAPTER EXERCISE-B



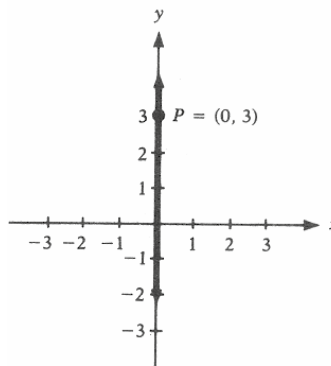
7.



8.



9.



10. $2x - y + 7 = 0$

12. $x - 2y + 5 = 0$

14. $x - 2y - 2 = 0$

16. $3x - y + 5 = 0$

18. $x - 4 = 0$

20. $x + y - 2 = 0$

11. $2x + 3y + 1 = 0$

13. $3x + y - 3 = 0$

15. $x - 1 = 0$

17. $2x - y = 0$

19. $x - 2y = 0$

IN-CHAPTER EXERCISE-C

$$1. \log_2 100 - \log_2 10 = \log_2 (10^2) - \log_2 (10) \\ = 2 \log_2 (10) - \log_2 (10) = \log_2 (10) = \frac{1}{\log_{10} (2)} = 3.3$$

$$2. \log(2ab/c^3) = \log 2 + \log a + \log b - 3 \log c$$

$$3. \frac{1}{2}[(2 \ln a - 2 \ln b) - 5 \ln c] = \frac{1}{2}[\ln a^2 - \ln b^2 - \ln c^5] \\ = \frac{1}{2} \left[\ln \left(\frac{a^2}{b^2} \right) - \ln(c^5) \right] = \ln \left(\frac{a^2}{b^2 c^5} \right)^{1/2}$$

$$4.(B) \text{ Let } \log_2 16 = n \\ \text{Then, } 2^n = 16 = 2^4 \Rightarrow n = 4 \therefore \log_2 16 = 4$$

$$5.(C) a^x = b^y \Rightarrow \log a^x = \log b^y \\ x \log a = y \log b$$

$$6.(A) \log_{10} \left(\frac{1}{70} \right) = \log_{10} 1 - \log_{10} 70 \\ = -(\log_{10} 7 + \log_{10} 10) = -(a + 1).$$

$$7.(B) \log_2 10 = \frac{1}{\log_{10} 2} = \frac{1}{0.3010} = \frac{10000}{3010} = \frac{1000}{301}$$

8.(B)

(A) Since $\log_a a = 1$, so $\log_{10} 10 = 1$.

$$(B) \log(2+3) = \log 5 \text{ and } \log(2 \times 3) = \log 6 = \log 2 + \log 3 \\ \therefore \log(2+3) \neq \log(2 \times 3)$$

(C) Since $\log_a 1 = 0$, so $\log_{10} 1 = 0$.

$$(D) \log(1+2+3) = \log 6 = \log(1 \times 2 \times 3) \\ = \log 1 + \log 2 + \log 3. \\ \text{So, (B) is incorrect.}$$

$$9.(C) \log_5 512 = \frac{\log 512}{\log 5} \\ = \frac{\log 2^9}{\log(10/2)} = \frac{9 \log 2}{\log 10 - \log 2} \\ = \frac{(9 \times 0.3010)}{1 - 0.3010} = \frac{2.709}{0.699} = \frac{2709}{699} = 3.876$$

$$10.(A) \log \frac{a}{b} + \log \frac{b}{a} = \log(a+b) \\ \Rightarrow \log(a+b) = \log \left\{ \left(\frac{a}{b} \right) \times \left(\frac{b}{a} \right) \right\} = \log 1 \\ \text{So, } a + b = 1.$$

IN-CHAPTER EXERCISE-D

1. Let $y = x^5 - 3x^3 + 1$
 $\frac{d}{dx}(y) = \frac{d}{dx}(x^5 - 3x^3 + 1) = 5x^4 - 9x^2$
2. $\frac{1}{2} \frac{d}{dx}(x^{10}) + \frac{1}{5} \frac{d}{dx}(x^5) + \frac{d}{dx}(6)$
 $= \frac{1}{2}(10x^9) + \frac{1}{5}(5x^4) + 0 = 5x^9 + x^4$
3. $y = t^8 - 2t^7 + 3t + 1$
 $\frac{dy}{dt} = 8t^7 - 14t^6 + 3$
4. $y = \frac{3}{x^2} + \frac{4}{x}$
 $y = 3x^{-2} + 4x^{-1}$
 $\frac{dy}{dx} = 3(-2)x^{-3} + 4(-1)x^{-2} = \frac{-6}{x^3} - \frac{4}{x^2}$
5. $y = 3x^{-2} - 7x^{-1} + 6$
 $\frac{dy}{dx} = 3(-2)x^{-3} - 7(-1)x^{-2} = \frac{-6}{x^3} + \frac{7}{x^2}$
6. $y = \frac{2}{5}x^{-1} - \frac{\sqrt{2}}{3}x^{-2}$
 $\frac{dy}{dx} = \frac{2}{5}(-1)x^{-2} - \frac{\sqrt{2}}{3}(-2)x^{-3} = -\frac{2}{5x^2} + \frac{2\sqrt{2}}{3x^3}$
7. $y = 3x^5 - x^2$
 $\frac{dy}{dx} = 15x^4 - 2x$
8. $\frac{d}{dx}\left[(x^2 + 3x)(x^3 - 9x)\right]$
 $= \frac{d}{dx}\left[x^5 + 3x^4 - 9x^3 - 27x^2\right]$
 $= 5x^4 + 12x^3 - 27x^2 - 54x$
9. Product Rule
 $= (x^3 - 8)\frac{d}{dx}\left(\frac{2}{x} - 1\right) + \left(\frac{2}{x} - 1\right)\frac{d}{dx}(x^3 - 8)$
 $= (x^3 - 8)\left(-\frac{2}{x^2}\right) + \left(\frac{2}{x} - 1\right)3x^2 = -2x + \frac{16}{x^2} + 6x - 3x^2$
10. $\frac{d}{dx}\left[\left(\frac{1}{x^2} + 3\right)\left(\frac{2}{x^3} + x\right)\right] = \frac{d}{dx}\left[\left(\frac{2}{x^5} + \frac{6}{x^3} + \frac{1}{x} + 3x\right)\right]$
 $= \frac{d}{dx}\left[2x^{-5} + 6x^{-3} + x^{-1} + 3x\right] = -\frac{10}{x^6} - \frac{18}{x^4} - \frac{1}{x^2} + 3$
11. π is a constant $\Rightarrow \frac{d\pi}{dx} = 0$
12. $y = (x-1)^2$
 $\frac{dy}{dx} = 2(x-1)$
13. $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$
 $\frac{dy}{dx} = -\frac{1}{2} \cdot x^{-\frac{3}{2}}$
14. $y = \frac{1}{x\sqrt{x}} = x^{-\frac{3}{2}}$
 $\frac{dy}{dx} = -\frac{3}{2}x^{-\frac{5}{2}}$
15. $y = ax^2 + bx + c$
 $\frac{dy}{dx} = 2ax + b$
16. $\frac{d}{dx}(\sin x + \cos x) = \cos x + (-\sin x)$
17. $\frac{d}{dx}(\sin^2 x) = \frac{d}{dx}(\sin x \times \sin x)$
 $= \sin x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(\sin x)$
 $= \sin x \cos x + \cos x \sin x = 2 \sin x \cos x = \sin 2x$
18. $\frac{d}{dx}(1 + \tan x) = 0 + \sec^2 x$
19. $\frac{d}{dx}(\ln x - e^x) = \frac{1}{x} - e^x$
20. $\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$

IN-CHAPTER EXERCISE-E

1. (i) $\int x^4 dx = \frac{x^5}{5} + C$
- (ii) $\int x^{5/4} dx = \frac{4}{9} x^{9/4} + C$
- (iii) $\int x^{-5} dx = \frac{-x^{-4}}{4} + C$
- (iv) $\int x^{-3/2} dx = -2x^{-1/2} + C$
- (v) $\int \frac{1}{\sqrt[3]{x^2}} dx = \int x^{-2/3} dx$
 $= \frac{x^{1/3}}{(1/3)} + c = 3x^{1/3} + c$
2. $\int (3x\sqrt{x} + 4\sqrt{x} + 5) dx = \frac{6}{5} x^{5/2} + \frac{8}{3} x^{3/2} + 5x + C$
3. $\int (ax^{5/2} + bx^{3/2} + cx^{1/2}) dx$
 $= \frac{2a}{7} x^{7/2} + \frac{2b}{5} x^{5/2} + \frac{2c}{3} x^{3/2} + C$
4. $\int \sqrt{x} \left(x^3 - \frac{2}{x} \right) dx = \int \left(x^{7/2} - 2x^{-1/2} \right) dx$
 $= \frac{x^{9/2}}{9/2} - 2 \frac{x^{1/2}}{1/2} + c = \frac{2}{9} x^{9/2} - 4x^{1/2} + c$
5. $\int \frac{1}{\sqrt{x}} \left(1 + \frac{1}{x} \right) dx = \int \left(x^{-1/2} + x^{-3/2} \right) dx$
 $dx = 2\sqrt{x} - \frac{2}{\sqrt{x}} + C$
6. $\int \frac{x^5 + x^{-2} + 2}{x^2} dx = \int \left(x^3 + x^{-4} + 2x^{-2} \right) dx$
 $dx = \frac{x^4}{4} - \frac{x^{-3}}{3} - 2x^{-1} + C$
7. $\int (3x^2 - 4x - 5) dx = x^3 - 2x^2 - 5x + C$
8. $\int \frac{x^3 - 1}{x - 1} dx = \int (x^2 + x + 1) dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C$
9. $\int \left(x^2 + 3x + \frac{1}{x^2} \right) dx = \frac{x^3}{3} + \frac{3x^2}{2} - \frac{1}{x} + C$
10. $\int (2 \sin x + 3 \cos x) dx = -2 \cos x + 3 \sin x + C$
11. $\int 2 \sin 35x dx = \frac{-2 \cos 35x}{35} + C$
12. $\int 3 \cos(8x - 1) dx = \frac{3 \sin(8x - 1)}{8} + C$
13. $\int \sec x (\tan x + \sec x) dx = \sec x + \tan x + C$
14. $\int_{-2}^0 (2x + 5) dx = \left[x^2 + 5x \right]_{-2}^0 = 6$
15. $\int_0^4 \left(3x - \frac{x^3}{4} \right) dx = \left(\frac{3x^2}{2} - \frac{x^4}{16} \right)_0^4 = 8$
16. $\int_0^1 (x^2 + \sqrt{x}) dx = \left(\frac{x^3}{3} + \frac{2x^{3/2}}{3} \right)_0^1 = 1$
17. $\int_1^{32} x^{-6/5} dx = \left(-5x^{-1/5} \right)_1^{32} = 2.5$
18. $\int_0^\pi \sin x dx = (-\cos x)_0^\pi = 2$
19. $\int_{\pi/2}^0 \frac{1 + \cos 2t}{2} dt = \left(\frac{t}{2} + \frac{\sin 2t}{4} \right)_{\pi/2}^0 = -\pi/4$
20. $\int_1^{-1} (r+1)^2 dr = \left(\frac{(r+1)^3}{3} \right)_1^{-1} = -8/3$

MISCELLANEOUS EXERCISE

1.(D) Let a and b the intercepts on the coordinate axes.

Given $a + b = -1$

$$\Rightarrow b = -a - 1 = -(a + 1)$$

The equation of the line is

$$x/a + y/b = 1$$

$$\Rightarrow \frac{x}{a} - \frac{y}{a+1} = 1 \quad \dots(i)$$

Since this line passes through $(4, 3)$,

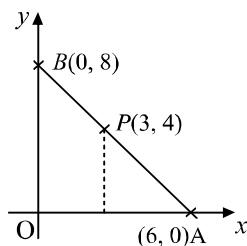
$$\frac{4}{a} - \frac{3}{a+1} = 1 \quad \Rightarrow \quad \frac{4a + 4 - 3a}{a(a+1)} = 1$$

$$\Rightarrow a + 4 = a^2 + a \quad \Rightarrow \quad a^2 = 4 \quad \Rightarrow \quad a = \pm 2$$

Therefore, the equation of the line [from (i)] is

$$\frac{x}{2} - \frac{y}{3} = 1 \quad \text{and} \quad \frac{x}{-2} + \frac{y}{1} = 1$$

2.(C) Since P is mid point of the line, intercept on axes are 6 and 8.



$$\therefore \text{Equation of line is } \frac{x}{6} + \frac{y}{8} = 1$$

$$\text{or } 4x + 3y = 24$$

3.(C) $\log_2 x = 10 \Rightarrow x = 2^{10}$. (C) is the correct answer

$$\therefore \log_x y = 100$$

$$y = x^{100}$$

$$y = (2^{10})^{100} \quad [\text{Put value of } x]$$

$$y = 2^{1000}$$

$$4.(D) \log_x \left(\frac{9}{16} \right) = -\frac{1}{2}$$

$$\Rightarrow x^{-1/2} = \frac{9}{16} \Rightarrow \frac{1}{\sqrt{x}} = \frac{9}{16}$$

$$\Rightarrow x = \left(\frac{16}{9} \right)^2 \Rightarrow x = \frac{256}{81}$$

5.(B) $y = e^x$ and $y = e^{-x}$ graphs are mirror image w.r.t. y -axis.

$$6.(B) \therefore v = \frac{ds}{dt} = \frac{d}{dt}(3t^2 + 4t + 5) = 6t + 4$$

$$7.(B) a = \frac{dv}{dt} = \frac{d}{dt}(3 \sin t) = 3 \frac{d}{dt} \sin t = 3 \cos t$$

$$8.(B) \text{Slope of tangent} = \frac{dy}{dx} = 2x = 2$$

$$\text{Equation } y - 1 = 2(x - 1)$$

$$y - 1 = 2x - 2$$

$$y = 2x - 1$$

$$9.(B) \int_{20}^{40} \frac{d}{10-x} = -1 [\ln(10-x)]_{20}^{40} \\ = -\ln\left(\frac{30}{10}\right) = -\ln(3)$$

$$10.(A) \frac{dy}{dx} = 2x + 1$$

$$\int dy = (2x + 1) dx \Rightarrow y = x^2 + x + c$$

$$\text{when } x = 1, y = 3 \text{ so } c = 1 \\ y = x^2 + x + 1$$

$$11.(B) \int du = - \int F dr = - \int \frac{k}{r^2} dr$$

$$u = \frac{k}{r} + u_0$$

$$12.(B) \text{Area} = \int_0^\pi \sin x dx = [-\cos x]_0^\pi = 1 - (-1) = 2$$